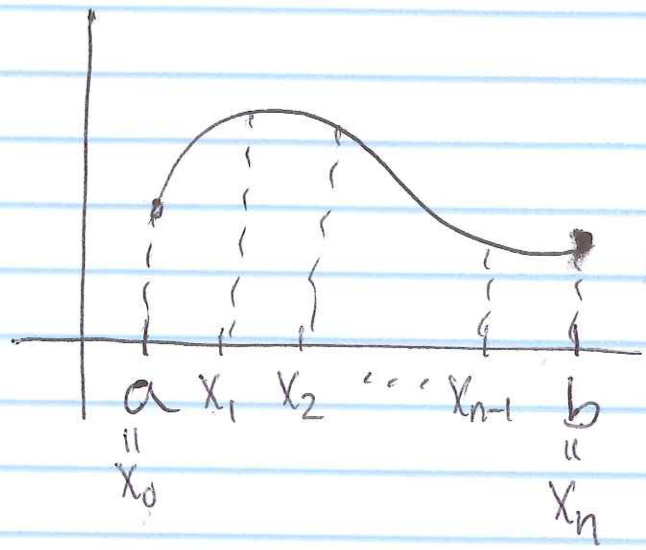
Riemann Sums:

How does sigma notation work?

A right Riemann Sum:

$$\underbrace{\sum_{i=1}^{n}}_{\text{start by letting } i=1 \text{, then } i=2 \text{, then } i=3 \;\vdots\; \text{stop when } i=n} \underbrace{f(x_i)\Delta x}_{\text{these are the terms that get added.}}$$



$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \dots + f(x_n)\Delta x$$

An Example: Area under $f(x) = x^2+1$ from -1 to 1

$$\Delta x = \frac{1-(-1)}{n} = \frac{2}{n}$$

in general $x_i = a + i\Delta x$

$$x_i = -1 + i\left(\frac{2}{n}\right) = -1 + \frac{2i}{n}$$

~~i | x_i | $f(x_i)$; 0 | -1 | $(-1)^2+1 = 2$~~

$$f(x_i) = \left(-1+\frac{2i}{n}\right)^2 + 1$$

Sum $\displaystyle\sum_{i=1}^{n}\left(\left(-1+\frac{2i}{n}\right)^2+1\right)\frac{2}{n}$

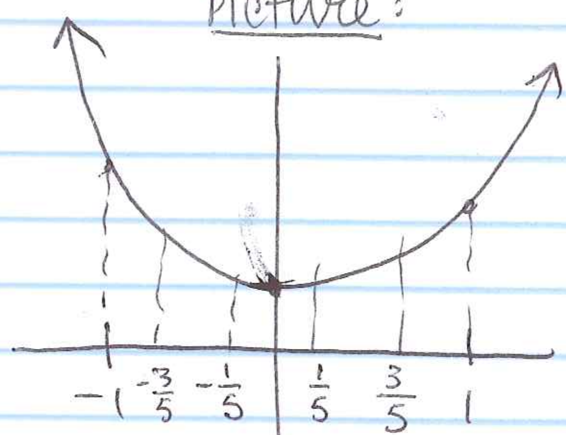
$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(-1+\frac{2i}{n}\right)^2+1\right)\frac{2}{n} = \int_{-1}^{1} (x^2+1)\,dx$$

<u>Some Sums:</u> Let's let $n=5$ first

i	x_i	$f(x_i)$
0	-1	2
1	$-3/5$	$34/25$
2	$-1/5$	$26/25$
3	$1/5$	$26/25$
4	$3/5$	$34/25$
5	1	2

$$\Delta x = \frac{1-(-1)}{5} = \frac{2}{5}$$

<u>Picture:</u>



Lower Sum:

$$\frac{2}{5}\left(f\left(-\frac{3}{5}\right)+f\left(-\frac{1}{5}\right)+f(0)+f\left(\frac{1}{5}\right)+f\left(\frac{3}{5}\right)\right) = \frac{2}{5}\left(\frac{34}{25}+\frac{26}{25}+1+\frac{26}{25}+\frac{34}{25}\right)$$

$$= 2.32$$

<u>Upper Sum:</u>

$$\frac{2}{5}\left(f(-1)+f\left(-\frac{3}{5}\right)+f\left(-\frac{1}{5}\right)+f\left(\frac{3}{5}\right)+f(1)\right)$$

$$= \frac{2}{5}\left(2+\frac{34}{25}+\frac{26}{25}+\frac{34}{25}+2\right) = 3.104$$

<u>Trapezoid:</u>

$$\frac{\left(\frac{2}{5}\right)}{2}\left(2+2\left(\frac{34}{25}\right)+2\left(\frac{26}{25}\right)+2\left(\frac{26}{25}\right)+2\left(\frac{34}{25}\right)+2\right)$$

$$= 2.72$$

Simpson's Method:

$$\frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\dots+4f(x_{n-1})+f(x_n)\right)$$

n must be even

Let $n=6$

$$\int_{-1}^{1} x^2+1\,dx \qquad \Delta x = \frac{2}{6} = \frac{1}{3}$$

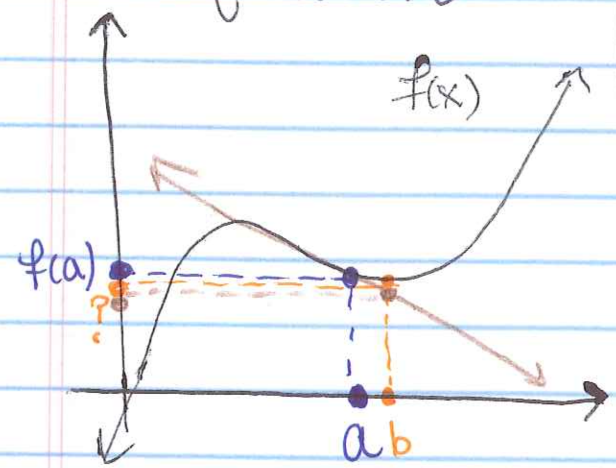
i	x_i	$f(x_i)$
0	-1	2
1	-2/3	13/9
2	-1/3	10/9
3	0	1
4	1/3	10/9
5	2/3	13/9
6	1	2

Simpson's: $\frac{1/3}{3}\left(2+4\left(\frac{13}{9}\right)+2\left(\frac{10}{9}\right)+4(1)+2\left(\frac{10}{9}\right)+4\left(\frac{13}{9}\right)+2\right)$

≈ 2.67

Linearization

Use: Estimate the value of a function



you know: values a and f(a)

you want: value of f(b) where b is "close" to a.

How: Find the tangent line to f(x) at x=a.

Plug in b to tangent line to approximate f(b).

Example: Find linearization of $\sqrt[3]{x}$ at $a=8$ and use it to approximate $\sqrt[3]{9}$

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$f(x)=\sqrt[3]{x}$

$f'(x)=\frac{1}{3x^{2/3}}$

$f'(8)=\frac{1}{3(8)^{2/3}}$

$=1/3(4)=\frac{1}{12}$

tangent line:

$y-2=\frac{1}{12}(x-8)$

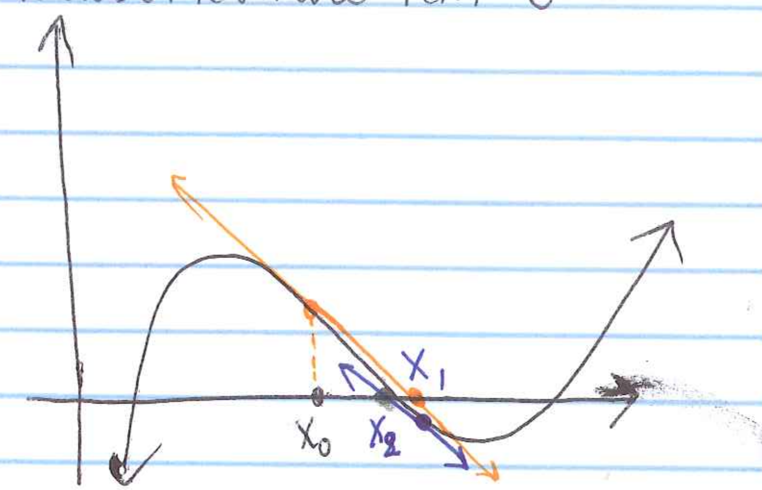
plug in 9, solve for y

$y=\frac{1}{12}+2=\frac{25}{12}\approx\sqrt[3]{9}$

$\frac{25}{12} = 2.0833$, $\sqrt[3]{9} \approx 2.08008$

Newton's Method

Use: Estimate roots of an equation values of x where f(x) = 0



Pick a value close to root. (x_0) draw tangent to f(x) at x_0 find where this intersects x-axis. This is x_1.

- Now draw tangent line at x_1 and find where that intersects x-axis. this is x_2.

In general $x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$

- the more times you do this, the closer your estimate will be.

Example: Use Newton's Method on $f(x)=x^3-9$ to estimate $\sqrt[3]{9}$ $f'(x)=3x^2$

i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
0	2	-1	12	25/12
1	25/12	$\left(\frac{25}{12}\right)^3-9$ = .04225	13.0208 = $3\left(\frac{25}{12}\right)^2$	$\frac{25}{12}-\frac{.04225}{13.0208} \approx 2.08009$

Ex: Find linearization of $\sqrt[3]{x}$ at $a=8$ and use it to approximate $\sqrt[3]{9}$

$f(x)=\sqrt[3]{x}$

$f'(x)=\frac{1}{3x^{2/3}}$

$f'(8)=\frac{1}{3(8)^{2/3}}=\frac{1}{12}$

tangent: $y-2=\frac{1}{12}(x-8)$

plug-in 9, solve for y

$y=\frac{1}{12}(9-8)+2=\frac{25}{12}\approx 2.0833$

Ex: Use Newton's Method on $f(x)=x^3-9$ to estimate a root of $f(x)$ (in this case the root will be at $\sqrt[3]{9}$) *use $x_0=2$

i	x_i	$f(x_i)$	$f'(x_i)$	$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$
0	2	−1	12	25/12
1	25/12	$\left(\frac{25}{12}\right)^3-9=$ .04225	$3\left(\frac{25}{12}\right)^2=$ 13.0208	2.08009

Arc Length:

Find arc length of $f(x) = 3x^2$ from 1 to 2.

So $f'(x) = 6x$

Arc length $= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

$= \int_1^2 \sqrt{1 + (6x)^2}\, dx$

$= \int_1^2 \sqrt{1 + 36x^2}\, dx$ ← No good way to integrate this. Let's approximate w/ trapezoids.

Let $n = 4$ $\quad \Delta x = \frac{2-1}{4} = \frac{1}{4}$

i	x_i	$f(x_i)$
0	1	$\sqrt{37}$
1	5/4	$\sqrt{1+36(5/4)^2} = \sqrt{57.25}$
2	6/4	$\sqrt{1+36(6/4)^2} = \sqrt{82}$
3	7/4	$\sqrt{1+36(7/4)^2} = \sqrt{111.25}$
4	2	$\sqrt{1+36(2)^2} = \sqrt{145}$

Estimate $= \frac{1/4}{2}\left(\sqrt{37} + 2\sqrt{57.25} + 2\sqrt{82} + 2\sqrt{111.25} + \sqrt{145}\right)$

≈ 9.0579

u-substitution

$$\int_0^1 2x^2\sqrt{1+x^3}\,dx$$

$u = 1+x^3$

$du = 3x^2\,dx$

$\frac{2}{3}du = 2x^2\,dx$

$$= \int_0^1 \sqrt{\underbrace{1+x^3}_{u}}\;\underbrace{2x^2\,dx}_{\frac{2}{3}du}$$

$$= \int_{1+0^3}^{1+1^3} \sqrt{u}\cdot\frac{2}{3}\,du$$

$$= \int_1^2 \frac{2}{3}\sqrt{u}\,du = \frac{2}{3}\cdot\frac{2}{3}u^{3/2}\Big|_1^2$$

$$= \boxed{\frac{4}{9}(2)^{3/2} - \frac{4}{9}}$$